Infinity

Exercises for Chapters 7 and 8 of Steinhart, E. (2009) *More Precisely: The Math You Need to Do Philosophy*. Broadview Press. (Version 1)

1. Explain why the idea of infinity has nothing to do with going on forever or trying to get there. Think about this carefully *before* you start writing.

To say that a set is infinite is to say that it contains a proper subset with exactly as many members as itself. No part of that definition says anything about time, or about anyone doing any activity to produce the set, or taking a journey to some far away place.

2. Explain why Zeus, if he accelerates, can write down all the finite numbers in 1 second even though there is no last finite number. Again, think about this carefully *before* you start writing.

Because for every number n, there is a time $(2^n-1)/2^n$ at which Zeus writes down n; and that time is less than 1. So every number is written down at some time < 1.

3. There are as many square numbers (1, 4, 9, 16, 25, etc.) as there are numbers. Why?

Because you can pair off n with n^2. The pairing is 1-1.

4. You love socks. On Monday you have one pair of socks for every natural number (there is a left sock and a right sock in each pair).

4.1 How is it possible to fit all these socks into a single finitely sized sock drawer in your dresser?

Make each next pair of socks half the size of the previous pair.

On Monday night, the devil destroys each one of your left socks, leaving you only with right socks. You discover this when you look at your socks on Tuesday morning.

4.2 True or false: On Tuesday, you have half as many socks as you did Monday. Explain your answer.

False. This is like the Devil destroying all the even numbers. So the odd numbers are left. But there are as many odds as there are numbers.

4.3 True or false: On Tuesday, you have exactly as many socks as you did Monday. Explain your answer.

False. This is like the Devil destroying all the even numbers. So the odd numbers are left. But there are as many odds as there are numbers.

5. You have a finitely sized urn filled with as many balls as natural numbers. Each ball has a number painted on it. You accelerate as follows: You pull out a ball in 1/2 second (this first ball has some number, but it need not be the number 1). Then you then pull out every next ball twice as fast. True or false: at 1 second, the urn is empty. Explain.

False. You might pull out just balls numbered with the even numbers, or some other proper but infinite subset of the natural numbers.

6. Use recursion to define an endless series. Give an initial rule, a successor rule, and a limit rule. To illustrate your example, feel free to use any type of object you like – angels, stars, universes, paradises, minds, weasels. Of course, you must state the relation between the objects (e.g. a successor number is greater than its predecessor; so a successor weasel is what? than its predecessor – and don't say more weasely).

There is an initial weasel. It's crazy.

For every crazy weasel, there's a successor weasel. It's twice as crazy.

For every endless series of crazy weasels, there's a limit weasel. The limit weasel is crazier than every weasel in the series of which it is the limit.

7. Explain why the Diagonal Argument suggests the Power Set Argument.

Because each infinite string of 0s and 1s defines a subset of the set of natural numbers. Going down the diagonal as Cantor says creates a the string corresponding to a subset that isn't in the original list of subsets.

8. Consider the binary tree partly shown in Figure 1. Each node in the tree (each point at which it branches) is labelled with either 0 or 1. The address of a node is the sequence of 0 or 1 branches that you take in traveling from the root on the right to the node. Thus the node whose address is 0000 is on the bottom right while the node whose address is 1111 is on the top right.

8.1 How many nodes are there in this tree? That is, what is the cardinal number of the set of nodes in the tree? Explain your answer.

There is 1 node at the root; then 2 nodes after the 1st branch; then 4 nodes. The number of nodes added with each branching is always finite. So you could just go through these nodes and number them all with finite numbers. There are as many nodes as there are finite numbers – and that's $\aleph_0$.

8.2 How many paths are there in this tree? That is, what is the cardinal number of the set of paths in the tree? Explain your answer.

Each path is an infinite string of 0s and 1s. Each of these strings corresponds uniquely to a subst of the natural numbers. So there are as many paths as there are subsets of the natural numbers. But there are uncountably many of those subsets. So the number of paths is uncountably infinite. It's Beth-1.

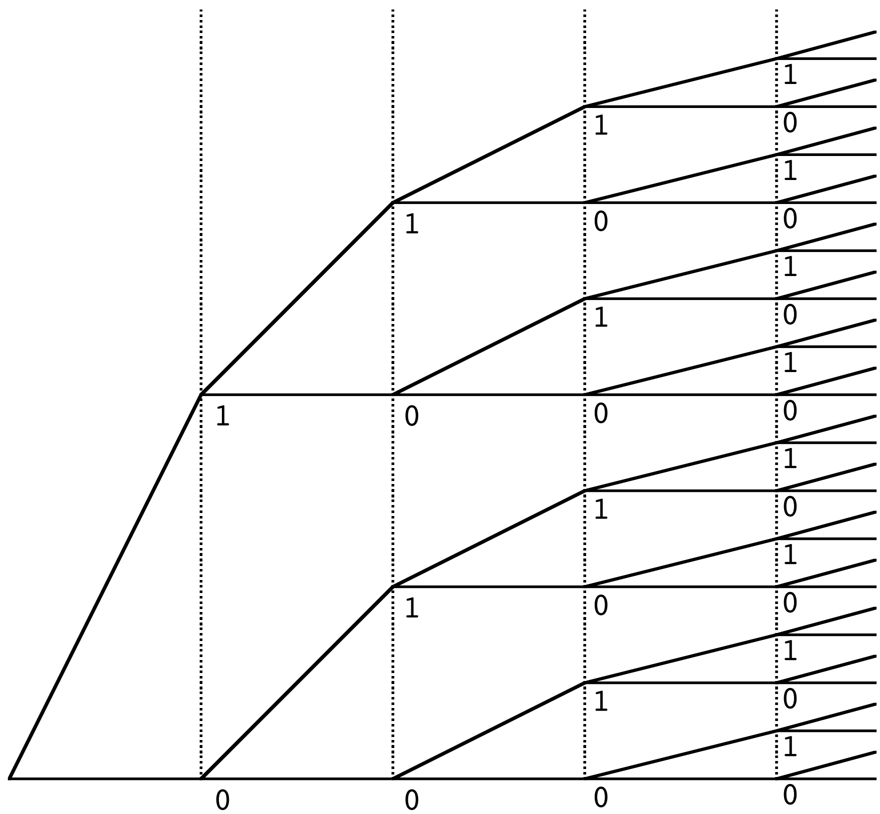

Figure 1. The first 4 iterations of an infinite binary tree.

9. Consider the collection of hereditarily finite sets. It's defined like this:

H(0) = {};

H(n+1) = pow H(n);

H = ∪{ H(n) | n is finite }.

9.1 How many sets are there in H? That is, what is the cardinal number of H?

There are $\aleph_0$. Same reasoning as for the number of nodes in the binary tree.

9.2 Is it possible to define a 1-1 correspondence between H and the natural numbers? If not, why not? If so, how would you do it?

Make a table with $\aleph_0$ many rows (this is a supertask). The rows are numbered with the natural numbers. Write {} in row 0. Now, for every *n*, look at H(*n*). You have to accelerate. For every new set added by H(*n*), write it in the next available row in the table. There are always only finitely many new sets, so each of these operations is finite. As you accelerate through all the H(*n*), you'll fill in all the rows in the table. The result is a 1-1 correspondence.