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THIS COMPREHENSIVE INDEX covers the terms and concepts of *An Introduction to Metalogic*, by Aladdin M. Yaqub, and was prepared by him.

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IS: If $\mathbf{X}[\mathbf{z}]$ is an AV formula in which \mathbf{z} is free, the following is a Peano Axiom:
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