Combinatorial Hierarchies

Supplement to Chapter 8 of Steinhart, E. (2009) *More Precisely: The Math You Need to Do Philosophy*. Broadview Press.

1. Levels of Existence

An *ontology* is a kind of catalog or dictionary that lays out categories of existing things. One way to organize an ontology is to sort things into levels. The levels are ordered by complexity. Simpler objects are on lower levels. Simpler objects combine to make more complex objects on higher levels. A system of levels is a *combinatorial hierarchy*. We will discuss a series of combinatorial hierarchies. Most of these hierarchies are developed by recent philosophers. A few are added just for continuity.

2. Rules for Making Combinatorial Hierarchies

2.1 Architecture of Combinatorial Hierarchies

Any combinatorial hierarchy has a general architecture:

- It has at least one *bottom* level of *individuals*.

- It has zero or more *intermediate* levels of *combinations*. A combination is formed when some number of simpler things are somehow unified to make a single composite or complex thing. Some writers will say that these combinations are *wholes*; others will say that they are collections – *sets* or *classes*.

- It has zero or one *top* levels of *unsurpassable combinations*. For most writers, these are the *proper classes*. Proper classes are collections that can't be members of more complex collections — after all, since they're at the top, there's nothing higher for them to be members of. They are unsurpassably general.

For any combinatorial hierarchy, there are three questions:

- What are the *individuals* on the bottom level?

- How *wide* is the hierarchy? Does each higher level include all possible combinations of objects on lower levels, or only some of those combinations?

- How *high* is the hierarchy? Does it have only finitely many levels or does it rise through endlessly many levels? Does it have a top system of proper classes?

2.2 Rules for Numbering the Levels

We sometimes use *ordinal numbers* to index the levels of combinatorial hierarchies. You're already familiar with the finite ordinals. These are just the natural numbers 0, 1, 2, 3, and so on. They are also known as the whole numbers or counting numbers. The series of ordinals continues far beyond the finite ordinals. We use Cantor's three number generating rules to introduce the series of ordinals:

1. *Initial Rule*. There is an initial ordinal 0. Every ordinal is a number, so 0 is a number.

2. *Successor Rule*. For every ordinal n, there exists a *successor ordinal* n+1 that is greater than n. This rule generates the finite ordinals 1, 2, 3, and so on.

3. *Limit Rule*. For any endlessly increasing series of ordinals, there exists a *limit ordinal* greater than every ordinal in the series. Since the series of finite ordinals 0, 1, 2, 3, . . . is endlessly increasing, there exists a limit ordinal ω greater than every finite ordinal. Since ω is greater than every finite ordinal, it is infinite.

2.3 Recursive Definition of the Levels of the Hierarchy

We can use the rules for making numbers to define levels. Here we go:

1. *Initial Rule*. A hierarchy has to have an *initial level*. The initial level includes all the individuals in the hierarchy. These are simples.

2. *Successor Rule*. A hierarchy has to have at least one *successor level*. Any successor level includes some combinations of objects from lower levels. These combinations are typically either *wholes* or *sets*. The rule that defines successor levels can be more or less open. It can allow only certain combinations or all combinations. A more open rule makes a wider hierarchy. The rule that defines successor levels can be restricted to only go to a certain height. Perhaps it does not go beyond the levels needed for science.

3. *Limit Rule*. A hierarchy may have one or more *limit levels*. A limit level is a level that exists above an endless series of levels. A limit level usually just contains all objects on all lower levels. The rule that defines limit levels can be restricted to only go to a certain height. Once again, it might not go beyond the levels needed by science.

4. *Final Rule*. A hierarchy may have a final level. For example, in class theories, this is the level of proper classes. They are collections that are too general to be sets. Proper classes don't enter into any further combinatorial relations. They aren't members of any larger sets or parts of any bigger wholes.

2.4 Writing the Levels Symbolically

We can adapt the rules for making numbers to make levels. Here we go:

1. *Initial Rule*. The initial level is V_0.

2. *Successor Rule*. A successor level comes after or above some previous level. If the previous level is the *n*-th level V_n, then the successor level is the (*n*+1)-th level V_{n+1}. Any successor level V_{n+1} is defined by a rule that describes how combinations are made from objects on lower levels.

3. *Limit Rule*. A limit level comes after or above an endless series of successor levels. If $V_0, V_1, V_2, \ldots$ is an endless series of levels, then the limit level above them all is the level V_ω where ω is the limit of the numbers 0, 1, 2, . . . Generally: a level V_L is any level whose subscript L is a limit number. V_L is all objects on all lower levels.

4. *Final Rule*. A final level is the level of maximal objects. For instance, in class theory, V is the *proper class* of all objects on all lower levels. Other proper classes exist alongside of V. They are subclasses of V. For example, while there is no set of ordinals, Ω is the proper class of all ordinals.

3. Van Inwagen's Vitalist Hierarchy

Some philosophers say that very few things exist. For example, van Inwagen (1995) argues for an ontology that has two types of things: physical simples and organisms. The simples are presumably the particles found in the Standard Model of Matter (e.g. the quarks and leptons). Simples are parts of organisms. So it's natural to regard these two types as two levels: (1) the level V_0 of physical simples and (2) the level V_1 of organisms. van Inwagen's ontology is a very short and narrow hierarchy. We can display it like this:

1. *Initial Rule*. The initial level V_0 is all physical simples.

2. *Successor Rule*. There is a single successor level above V_0. It is the level V_1. V_1 contains all living combinations of objects on the lower level V_0. There are no levels above V_1. Hence living things have no internal structure.

3. *Limit Rule*. There are no limit levels.

4. *Final Rule*. There is no final level.

What's missing? Well, if you agree with van Inwagen's reasoning, then nothing is missing! But other authors have developed richer hierarchies. The *most generous point of view* considers all the objects posited in all combinatorial hierarchies. From the most generous point of view, the following kinds of objects are missing:

- The empty combination {} is missing. This combination is a set, not a whole.

- Pure combinations are missing. These are sets built from the empty set. Examples include sets like {{}} and {{}, {{}}}.

- Unit combinations are missing. For any simple A, the unit combination {A} is absent. This combination is a set, not a whole.

- Most plural combinations are missing. Specifically, all non-living plural combinations are missing. There are no complex particles like protons or neutrons. There are no atoms or molecules. These combinations are wholes.

- Structured combinations are missing. A structured combination has internal braces that denote internal structural boundaries. For instance, {{A, B}, {C, D}}. These combinations can be regarded as wholes so long as they are closed. A closed combination is one that behaves like a container.

- Open combinations are missing. Open combinations are those that don't behave like containers. They violate the notion that once a thing is inside a combination, it's no longer available for further combination. For instance, {A, {A, B}} is an open combination. You can't have a container that has A and B in it like that.

- Infinitely complex combinations are missing.

- Non-actual combinations are missing. These are combinations that involve non-actual objects (e.g. the inhabitants of other non-actual possible universes).

- Large cardinals are missing. Hence the levels they index are missing. Large cardinals are numbers that can't be derived by an combinatorial operations based on ω. They have to be asserted with special axioms.

- Proper classes are missing. Proper classes are too general to be sets. For example, there is no set of all sets. But there can be a proper class of all sets. There is no set of all ordinal numbers. But there can be a proper class of all ordinal numbers.

4. Goodman's Nominalist Hierarchy

van Inwagen says that physical simples and organisms are the only physical things (1995: 98) He says that complex inanimate things (like rocks and tables) are mere "virtual objects" (1995: 112). He says these virtual objects don't really exist. Of course, many philosophers believe that rocks and tables exist. They posit a richer level above V_0.

A *nominalist* like Nelson Goodman (1965) posits a hierarchy that is very low and very narrow. Here's how a nominalist like Goodman would answer the three hierarchy questions:

- What are the *individuals* on the bottom level? The individuals are actual simple physical things. We can stick with the particles in the Standard Model of Matter.

- How *wide* is the hierarchy? The next level contains all *plural* combinations. Plural combinations are *wholes*. Every whole has more than one individual. There is no empty combination (a conbination with zero individuals). There are no unit combinations (with exactly one individual). So the width of the hierarchy is very restricted.

- How *high* is the hierarchy? Sums have no internal structure. You can erase any internal braces in a sum. Example: the sum {{A, B}, {C, D}} = {A, B, C, D}. Hence there are no levels above V_1.

These rules define the architecture of Goodman's nominalist hierarchy:

1. *Initial Rule*. The initial level V_0 is all actual physical simples.

2. *Successor Rule*. There is a single successor level above V_0. It is the level V_1. V_1 contains all *plural* combinations of objects on the lower level V_0. These combinations are wholes. There are no levels above V_1. Wholes have no internal structure.

3. *Limit Rule*. There are no limit levels.

4. *Final Rule*. There is no final level.

Table 1 illustrates Goodman's nominalist hierarchy. The bottom level of simples contains just three simples A, B, and C. The next level contains wholes formed from them. Note that the combinations {}, {A}, {B}, and {C} are missing on level V_1.

Level	Objects on Level
V_1	{A, B} {A, C} {B, C} {A, B, C}
V_0	A, B, C

Table 1. A sample nominalist hierarchy.

What's missing? From the most generous point of view, the following kinds of objects are missing from Goodman's nominalist hierarchy:

- The empty combination is missing.
- Pure combinations are missing.
- Unit combinations are missing.
- Structured combinations are missing.
- Open combinations are missing.
- Infinitely complex combinations are missing.
- Non-actual combinations are missing.
- Large cardinals and their levels are missing.
- Proper classes are missing.

5. The Rich Mereological Hierarchy

Mereology is the study of the part-whole relation. Goodman's nominalist hierarchy is a mereological hierarachy. But the combinatorial principles used to form wholes in Goodman's hierarchy are severly restricted. It's possible to lift these restrictions. A *rich mereological hierarchy* is somewhat wider and higher than the poor mereological hierarchy. We're not aware of any author who's written explicitly about rich mereological hierarchies. However, rich mereological hierarchies are a stage on the way to hierarchies that many authors have developed. So it's worth looking at the structure of rich mereological hierarchies. Here's how rich mereology answers the hierarchy questions:

- What are the *individuals* on the bottom level? As before, these are actual simple physical things – the particles in the Standard Model.

- How *wide* is the hierarchy? The next level contains all *closed* combinations. These are wholes. You can think of them in terms of putting things in bags. You can take A and B and put them into a bag to make {A, B}. You can put that bag and the individual C into another bag to make {{A, B}, C}. But once an object is in a bag, you can't take it out to put it in another bag. You can't make the combination {A, {A, B}}. To make that combination, you'd have to take A out of {A, B}. But the rules say that you can't. Since these combinations are based on the notion of enclosing things in containers, we say they're closed. There are still no empty or unit combinations.

- How *high* is the hierarchy? Rich wholes have internal structure. For instance, the rich whole {{A, B}, {C, D}} is *not* identical with {A, B, C, D}. Hence there can be levels above the first level V_1. We're still trying to stay as close to concrete reality as possible. So we won't posit any infinitely complex objects. Hence no limit levels.

These rules define the architecture of the rich mereological hierarchy:

1. *Initial Rule*. The initial level V_0 is all actual physical simples.

2. *Successor Rule*. For every finite number n, there exists a successor level V_{n+1}. The successor level V_{n+1} is the collection of all the closed combinations formed from objects at lower levels.

3. *Limit Rule*. There are no limit levels.

4. *Final Rule*. There is no final level.

Table 2 shows a sample rich mereological hierarchy. The hierarchy in Table 2 is a part-whole hierarchy – objects on level n are *parts* of objects on level n+1.

Level	Kind of Material Thing
V_{10}	Societies & ecosystems.
V_9	Organisms.
V_8	Organs, tissues.
V_7	Cells.
V_6	Organelles.
V_5	Molecular assemblies.
V_4	Molecules.
V_3	Atoms.
V_2	Atomic nuclei.
V_1	Protons, neutrons, electron shells.
V_0	Quarks, leptons, bosons.

Table 2. A sample rich mereological hierarchy.

What’s missing? From the most generous point of view, the following kinds of objects are missing from the poor mereological hierarchy:

- The empty combination is missing.
- Pure combinations are missing.
- Unit combinations are missing.
- Open combinations are missing.
- Infinitely complex combinations are missing.
- Non-actual combinations are missing.
- Large cardinals and their levels are missing.
- Proper classes are missing.

6. Maddy’s Pluralist Hierarchy

A combinatorial pluralist posits a hierarchy that is higher and wider than the mereological hierarchy. The pluralist allows all plural combinations – hence the pluralist goes beyond mereology and enters into set theory. The objects in the pluralist hierarchy are not wholes – they are sets. Here’s how a pluralist would answer the three hierarchy questions:

- What are the *individuals* on the bottom level? The individuals are actual simple physical things.
- How *wide* is the hierarchy? The next level contains all plural combinations – both closed and open. Since open combinations are not wholes, the pluralist hierarchy is not mereological. It is a set theoretic hierarchy.
- How *high* is the hierarchy? The hierarchy rises through all the levels that are needed for science; if science needs infinitely complex objects, then the hierarchy rises through inifinite levels. These are limit levels. Every limit level contains all objects on all lower levels. Does it have a top level of proper classes? Certainly, the pluralist can define a top level of proper classes. But it seems better to wait for more purely set theoretic hierarchies before we add one. So we won’t give the pluralist proper classes.

For example: Maddy (1992: 156 - 157) says V_{n+1} is all plural combinations of objects from all lower levels. Maddy seems to allow infinite sets that are used for science. See Table 3 for an illustration. The rules of Maddy's actualist hierarchy are:

1. *Initial Rule*. The initial level V_0 is all actual physical simples.

2. *Successor Rule*. For every successor number *n*+1 less than the biggest infinity needed for actual science, there is a successor level V_{n+1}. Each successor level contains all *plural* combinations of all objects on all lower levels. Since these can include open combinations like {A, {A, B}}, these combinations are *sets* rather than wholes.

3. *Limit Rule*. For every limit number L less than the biggest infinity needed for actual science, there is a successor level V_L. Each limit level contains all objects from all lower levels. It's just a big bag that collects combinations.

4. *Final Rule*. There is no final level. Of course, you might say that the final level is the level indexed by the biggest infinity needed for actual science. And while that's true, that level is a limit level. It isn't a different kind of level.

Level	Some Objects on Level
. . .	. . .
V_3	{A, B, {A, B, {A, B}}}
V_2	{A, B, {A, B}} {{A, C}, {B, C}} {C, {A, B, C}}
V_1	{A, B} {A, C} {B, C} {A, B, C}
V_0	A, B, C

Table 3. A sample pluralist hierarchy.

What's missing? From a purely logical or formal point of view, the following kinds of objects are missing from the poor mereological hierarchy:

- The empty combination is missing.
- Pure combinations are missing.
- Unit combinations are missing.
- Non-actual combinations are missing.
- Large cardinals and their levels are missing.

- Proper classes are missing.

7. The ZFCU Hierarchy

We now move to a generalization of Maddy's pluralist hierarchy. The generalization allows us to say that V_{n+1} is *all* combinations of all objects on all lower levels. The result is a richer set theoretic hierarchy. This hierarchy is defined by the axioms of standard Zermelo – Fraenkel – Choice set theory plus an axiom that says there are some individuals on the bottom level. In set theory, these individuals are known as urelemente. So this hierarchy is the ZFC hierarchy plus urelemente (see McGee, 1997).

We'll refer to this as the *ZFCU hierarchy*. Here's how an advocate of the ZFCU hierarchy would answer the three hierarchy questions:

- What are the *individuals* on the bottom level? The individuals are actual simple physical things.

- How *wide* is the hierarchy? Every next level contains all possible combinations. There are no restrictions. The combinations in the ZFCU hierarchy are sets.

- How *high* is the hierarchy? ZFC defines an ordinal number line. This line includes all the finite ordinals (the natural numbers). It also includes a long sequence of infinite ordinal numbers. For any number that is on the ZFC ordinal line, there is a level of the ZFCU hierarchy that is indexed by that number. However, ZFC does not define any proper classes at the top of the hierarchy.

The ZFCU hierarchy has all the sets needed for science. It has all the sets needed for most (but not all) logical and mathematical theories. See Table 4 for a simple illustration. More formally, the ZFCU hierarchy is defined by these rules:

1. *Initial Rule*. The initial level V_0 is all actual physical simples.

2. *Successor Rule*. For every successor number n+1 on the ZFC ordinal number line, there is a successor level V_{n+1}. Each successor level contains *all* combinations of all objects on all lower levels. It includes both empty and unit combinations. The empty set appears on level V_1. Above that level, pure sets begin to form and become more complex. The width of the ZFCU hierchy is unsurpassable.

3. *Limit Rule*. For every limit number L on the ZFC ordinal number line, there is a successor level V_L. Each limit level contains all objects from all lower levels. It's just a big bag that collects combinations.

4. *Final Rule*. There is no final level.

Level	Some Objects on Level
. . .	. . .
V_3	$\{\{\},\{\{\}\}\}$ $\{\{\{\}\}\}$ $\{\{\{A\}\}\}$ $\{A,\{A\},\{\{A\}\}\}$ $\{B,\{\{B\},\{C\}\}\}$
V_2	$\{\{\}\}$ $\{\{A\}\}$ $\{\{\},\{A\}\}$ $\{A,\{A\}\}$ $\{\{B\},\{C\}\}$
V_1	$\{\}$ $\{A\}$ $\{B\}$ $\{C\}$ $\{A,B\}$ $\{A,C\}$ $\{B,C\}$ $\{A,B,C\}$
V_0	A, B, C

Table 4. A small part of the ZFCU hierarchy.

What's missing? From the most generous point of view, the following kinds of objects are missing from ZFCU hierarchy:

- Non-actual combinations are missing.
- Large cardinals and their levels are missing.
- Proper classes are missing.

8. Lewis's Possibilist Hierarchy

A *modal realist* posits a bottom level that contains all *possible* individuals. Some of these possible individuals are actual physical simples; others are non-actual physical simples. They are the building blocks of physical structures in non-actual universes. So much for the bottom level. For consistency, a modal realist ought to posit all *possible* higher levels involving all *possible* combinations of lower-level objects. At least at some times, David Lewis is such a modal realist. His ontology "consists of possibilia – particular, individual things, some of which comprise our actual world and others of which are unactualized – together with the iterative hierarchy of classes built up from them" (1983: 9).

A *modal realist* posits a hierarchy that is higher and wider than the actualist hierarchy. Here's how a modal realist (like Lewis) would answer the three hierarchy questions:

- What are the *individuals* on the bottom level? The individuals are all possible physical simples or all possible physical things. Some of these are in our universe, while others are in other physical universes.
- How *wide* is the hierarchy? The next level contains all possible combinations of objects on lower levels. Limit levels contain all objects on all lower levels.

- How *high* is the hierarchy? The hierarchy rises through all the levels that are needed for all *possible* scientific theories. It's not clear how high this is, but at this point it seems reasonable to stick with the height of the ZFC hierarchy.

One good candidate for the possibilist hierarchy is known as the Von Neumann - Bernays hierarchy (the VNB hierarchy; see Hamilton, 1982: ch. 4; Devlin, 1991: ch. 2). It is defined by these four rules:

1. *Initial Rule*. The initial level V_0 is all possible physical simples.

2. *Successor Rule*. For every successor number n+1 less than the biggest infinity needed for any possible science, there is a successor level V_{n+1}. Each successor level contains *all* combinations of all objects on all lower levels.

3. *Limit Rule*. For every limit number L less than the biggest infinity needed for possible science, there is a successor level V_L. Each limit level contains all objects from all lower levels. It's just a big bag that collects combinations.

4. *Final Rule*. There are no proper classes.

What's missing? From the most generous point of view, the following kinds of objects are missing from the Lewisian hierarchy:

- Large cardinals and their levels are missing.

- Proper classes are missing.

9. Quine's Hierarchy of Pure Sets

Many writers have observed that physical things have exact representatives in the hierarchy of pure sets (Harman, 1967; Gottlieb, 1976; Tegmark, 1998). If space-time is 4D, continuous, and Eucldean, then space-time is exactly represented by the set of quads of real numbers. So space-time is exactly represented by R^4. If things are individuated by their space-time locations, then they are exactly represented by functions from R^4 to {0, 1}. The gravitational force acting at a point is a vector. It is a triple of real numbers. The gravitational field is exactly represented by a function from R^4 to R^3.

Since physical things have exact representatives in the hierarchy of pure sets, it seems redundant to include physical things in the bottom level of the hierarchy. Any physical thing can be replaced with one of its representatives. We can apply Occam's Razor to remove them. And indeed, Quine argued frequently that we ought to remove them (see Quine, 1969: 147 - 152; 1976; 1978; 1981: 15 – 18). Quine's ontology is therefore an ontology of pure sets: all that exists is the hierarchy of pure sets. The bottom level V_0 is empty. All possible physical things will appear as pure classes on higher levels.

However, the height of the Quinean hierarchy of pure sets is limited by the needs of science. Quine says: "So much of mathematics as is wanted for use in empirical science is for me on a par with the rest of science. [Some advanced set theory is] on the same footing insofar as [it comes] to a simplificatory rounding out, but anything further is on a par with uninterpreted systems" (1984: 788; quoted from Shapiro, 2000: 219 - 220). Here's how Quine would answer the three hierarchy questions:

- What are the *individuals* on the bottom level? There are no objects on the bottom level. The bottom level is the empty set. Since all other objects are built from the empty set, all other objects are sets of sets. They are all *pure* sets.

- How *wide* is the hierarchy? The next level contains all possible combinations of objects on lower levels. Limit levels contain all objects on all lower levels. The hierarchy is as wide as logically possible.

- How *high* is the hierarchy? The hierarchy rises through al the levels needed for science plus levels that round it out mathematically. The natural way to define this is to say that the Quinean hierarchy is as high as the standard ZFC hierarchy. For the sake of logic, we'll extend the Quinean hierarchy to include a proper class of all sets V. The universal quantifier refers to this proper class. Thus $\forall$ refers to V.

To formalize the Quinean hierarchy, we need to extend the ZFC axioms to include proper classes. While there are several ways to do this, many mathematicians today use an axiom system known as the Von Neumann – Bernays system (see Hamilton, 1982: ch. 4; Devlin, 1991: ch. 2). We can refer to this axiom system as VNB. VNB is just ZFC rewritten to allow for proper classes. It defines an ordinal number line. We'll use that ordinal number line to index the levels of the Quinean hierarchy. Thus

1. *Initial Rule*. The initial level V_0 is the empty set {}.

2. *Successor Rule*. For every successor number n+1 on the VNB ordinal number line, there is a successor level V_{n+1}. Each successor level contains *all* combinations of all objects on all lower levels.

3. *Limit Rule*. For every limit number L on the VNB ordinal number line, there is a successor level V_L. Each limit level contains all objects from all lower levels. It's just a big bag that collects combinations.

4. *Final Rule*. The proper class V contains all objects on all lower levels.

What's missing? From the most generous point of view, the following kinds of objects are missing from the Quinean hierarchy:

- Large cardinals and the levels they index are missing.

10. The Pythagorean Hierarchy

We arrive at last at a hierarchy whose combinatorial richness is restricted only by logical consistency. It is that hierarchy than which none richer is logically possible. It is *combinatorially unsurpassable*. This hierarchy is a *plenum*. It satisfies the *Principle of Plenitude* (Kane, 1976, 1986; Bricker, 1991; Balaguer, 1998b). An advocate of this hierarchy argues that all things, whether actual or possible, physical or mathematical, are just pure sets. This is a kind of Pythagoreanism. So we'll refer to this as the Pythagorean hierarchy. A Pythagorean answers the three hierarchy questions like this:

- What are the *individuals* on the bottom level? There are no individuals on the bottom level. The bottom level is empty.

- How *wide* is the hierarchy? The next level contains all possible combinations of objects on lower levels. Limit levels contain all objects on all lower levels. The hierarchy is as wide as logically possible. It is unsurpassably wide.

- How *high* is the hierarchy? The hierarchy rises through all logically possible levels. It has levels indexed by all consistent large cardinals (see Drake, 1974). It has a level of proper classes. It is unsurpassably high.

We say VNB* is VNB plus axioms for all consistently definable large cardinals. VNB* defines that ordinal number line than which no longer is logically possible. We'll refer to this as the *Long Line*. For every *k* on the Long Line, the Pythagorean hierarchy has a level indexed by *k*. Hence these rules formally define the Pythagorean hierarchy:

1. *Initial Rule*. The initial level V_0 is the empty set {}.

2. *Successor Rule*. For every successor number n+1 on the Long Line, there is a successor level V_{n+1}. Each successor level contains *all* combinations of all objects on all lower levels. Thus $V_{n+1} = \text{pow } V_n$.

3. *Limit Rule*. For every limit number L on the Long Line, there is a limit level V_L. Each limit level contains all objects from all lower levels. Thus $V_L = \cup\{ V_n \mid n < L\}$.

4. *Final Rule*. The proper class V contains all objects on all lower levels. The proper class Ω contains all ordinals on the Long Line. Thus $V = \cup\{ V_n \mid n < \Omega\}$.

References

Bunge, M. (1966) On null individuals. *Journal of Philosophy 63* (24), 776 - 778.

Carnap, R. (1958) *Meaning and Necessity* (2nd Ed.). Chicago: University of Chicago Press.

Chihara, C., Lin, Y. & Schaffter, T. (1975) A formalization of nominalistic set theory. *Journal of Philosophical Logic 4,* 155 - 140.

Colyvan, M. (2001) *The Indispensability of Mathematics*. New York: Oxford University Press.

Goodman, N. (1956) A world of individuals. In I. M. Bochenski, A. Church, N. Goodman (Eds.), *The Problem of Universals*. Notre Dame: University of Notre Dame Press, 13 - 32.

Gottlieb, D. (1976) Ontological reduction. *Journal of Philosophy 73* (3), 57 - 76.

Harman, G. (1967) Quine on meaning and existence II. *Review of Metaphysics 21* (2), 343 - 367.

Lewis, D. (1970) Nominalistic set theory. *Nous 4* (3), 225 - 241.

Lewis, D. (1983) New work for a theory of universals. In D. Lewis (Ed.) (1999) *Papers in Metaphysics and Epistemology*. New York: Cambridge University Press, 8 – 55.

Maddy, P. (1992) *Realism in Mathematics*. New York: Oxford University Press.

Martin, R. M. (1943) A homogeneous system for formal logic. *Journal of Symbolic Logic 8,* 1 - 27.

Martin, R. M. (1965) Of time and the null individual. *Journal of Philosophy 62* (24), 723 - 736.

McGee, V. (1997) How we learn mathematical language. The Philosophical Review 106 (1), 35 – 68.

Quine, W. V. (1969) *Ontological Relativity and Other Essays*. New York: Columbia University Press.

Quine, W. V. (1976) Wither physical objects? *Boston Studies in the Philosophy of Science 39,* 497 - 504.

Quine, W. V. (1978) Facts of the matter. *Southwestern Journal of Philosophy 9* (2), 155 - 169.

Quine, W. V. (1981) *Theories and Things*. Cambridge, MA: Harvard University Press.

Quine, W. V. O. (1953) Two dogmas of empiricism. In A. P. Martinich (Ed.) (1992) *The Philosophy of Language*. New York: Oxford University Press, 26-39.

Quine, W. V. O. (1984) Review of Parsons' *Mathematics in Philosophy*. *Journal of Philosophy 81,* 783 - 794.

Rosenberg, J. (1970) Notes on Goodman's nominalism. *Philosophical Studies 21* (1-2), 19 - 24.

Rucker, R. (1995) *Infinity and the Mind*. Princeton, NJ: Princeton University Press.

Shapiro, S. (2000) *Thinking about Mathematics*. New York: Oxford University Press.

Tegmark, M. (1998) Is ‘the Theory of Everything’ merely the ultimate ensemble theory? *Annals of Physics 270,* 1 – 51.

Tegmark, M. (2003) Parallel universes. *Scientific American 288* (5), 40 – 51.